

Differential Equations and Hecke Operators

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Abstract—Certain classes of linear ordinary differential equations are closely linked to automorphic forms of one variable. We construct Hecke operators on the space of certain nonhomogeneous linear ordinary differential equations that are compatible with the usual Hecke operators on automorphic forms.

Keywords—Differential equations, Hecke operators, Automorphic forms.

1. INTRODUCTION

Automorphic forms play a major role in number theory, and they are also related to various other areas of mathematics. Close ties between automorphic forms and a certain class of linear ordinary differential equations have been known since the nineteenth century (cf. [1,2]). In this paper we consider one aspect of such a connection.

Hecke operators play an important role in the theory of automorphic forms. They are certain endomorphisms of the space of various automorphic forms and are used mainly as a tool for the investigation of multiplicative properties of Fourier coefficients of such automorphic forms (see, e.g., [3,4]). Certain sets of Hecke operators have ring or algebra structures thus producing Hecke rings or Hecke algebras (cf. [5]). Various forms of Hecke algebras have been studied extensively over the years in connection with several branches of mathematics that range from the theory of finite group representations to the theory of knots and quantum groups.

The purpose of this paper is to construct Hecke operators on the space of certain nonhomogeneous linear ordinary differential equations that are compatible with the usual Hecke operators on automorphic forms with respect to the connection of such differential equations with automorphic forms.

2. AUTOMORPHIC FORMS AND HECKE OPERATORS

In this section we review automorphic forms and Hecke operators acting on the space of automorphic forms (see, e.g., [3,6] for details). Let $GL^+(2, \mathbb{R})$ (respectively, $SL(2, \mathbb{R})$) be the multiplicative group of 2×2 real matrices of positive determinant (respectively, determinant one). Then $GL^+(2, \mathbb{R})$ and $SL(2, \mathbb{R})$ act on the Poincaré upper half plane

$$\mathcal{H} = \{z \in \mathbb{C} \mid \text{Im } z > 0\}$$

by linear fractional transformations. If $\alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is an element of $GL^+(2, \mathbb{R})$ and if $f : \mathcal{H} \rightarrow \mathbb{C}$ is a function, then we set $j(\alpha, z) = cz + d$ and

$$(f|_k \alpha)(z) = \det(\alpha)^{k/2} j(\alpha, z)^{-k} f(\alpha z) \quad (1)$$

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for $z \in \mathcal{H}$ and $k \in \mathbb{Z}$. Let $\Gamma \subset SL(2, \mathbb{R})$ be a Fuchsian group of the first kind, so that the quotient space $\Gamma \backslash \mathcal{H} \cup \{\text{cusps}\}$ is a compact Riemann surface, and let k be a nonnegative integer.

DEFINITION 2.1. *A meromorphic function $f : \mathcal{H} \rightarrow \mathbb{C}$ is an automorphic form of weight k for Γ if it satisfies*

$$(f|_k \gamma)(z) = f(z)$$

for all $\gamma \in \Gamma$ and $z \in \mathcal{H}$ and is meromorphic at the cusps. We shall denote by $\mathcal{A}_k(\Gamma)$ the space of all automorphic forms of weight k for Γ .

If Γ_1 and Γ_2 are subgroups of $GL^+(2, \mathbb{R})$, then we say that Γ_1 and Γ_2 are *commensurable* and write $\Gamma_1 \sim \Gamma_2$ if $\Gamma_1 \cap \Gamma_2$ has finite index in both Γ_1 and Γ_2 . Let Γ be the Fuchsian group of the first kind as above, and set

$$\tilde{\Gamma} = \{g \in GL^+(2, \mathbb{R}) \mid g\Gamma g^{-1} \sim \Gamma\}.$$

If $\alpha \in \tilde{\Gamma}$, then the double coset $\Gamma\alpha\Gamma$ has a decomposition of the form

$$\Gamma\alpha\Gamma = \coprod_{\nu=1}^d \Gamma\alpha_\nu \quad (2)$$

for some $\alpha_\nu \in GL^+(2, \mathbb{R})$, $\nu = 1, \dots, d$. Then the Hecke operator on $\mathcal{A}_k(\Gamma)$ associated to $\alpha \in \tilde{\Gamma}$ is the map $T(\alpha) : \mathcal{A}_k(\Gamma) \rightarrow \mathcal{A}_k(\Gamma)$ defined by

$$T(\alpha)f = \det(\alpha)^{k/2-1} \sum_{\nu=1}^d (f|_k \alpha_\nu) \quad (3)$$

for all $f \in \mathcal{A}_k(\Gamma)$.

3. DIFFERENTIAL EQUATIONS

Let $\Gamma \subset SL(2, \mathbb{R})$ be a Fuchsian group of the first kind that does not contain elements of finite order. Then the quotient $X = \Gamma \backslash \mathcal{H}^*$ is a compact Riemann surface, where \mathcal{H}^* is the union of \mathcal{H} and the cusps of Γ , and it can also be regarded as an algebraic curve over \mathbb{C} . Let x be a nonconstant element of the function field $K(X)$ of X , and let $\varphi : \mathcal{H} \rightarrow \mathbb{C}$ be a nonzero automorphic form of weight one for Γ . Then φ satisfies a second order linear differential equation $\Lambda_X^2 f = 0$ with

$$\Lambda_X^2 = \frac{d^2}{dx^2} + P_X(x) \frac{d}{dx} + Q_X(x) \quad (4)$$

that has regular singular points, where $P_X(x)$ and $Q_X(x)$ are elements of $K(X)$. By pulling back (4) via the natural projection $\mathcal{H}^* \rightarrow X = \Gamma \backslash \mathcal{H}^*$, we obtain a differential operator

$$\Lambda^2 = \frac{d^2}{dz^2} + P(z) \frac{d}{dz} + Q(z)$$

such that $P(z)$ and $Q(z)$ are meromorphic functions on \mathcal{H}^* , the functions $\omega_1(z) = z\varphi(z)$ and $\omega_2(z) = \varphi(z)$ for $z \in \mathcal{H}$ are linearly independent solutions of the associated homogeneous equation $\Lambda^2 f = 0$, and the regular singular points of Λ^2 coincide with the cusps of Γ (see [7] for details). Given a positive integer m let $S^m(\Lambda^2)$ be the linear ordinary differential operator of order $m+1$ such that the solutions of the corresponding homogeneous equation $S^m(\Lambda^2)f = 0$ are of the form

$$\sum_{i=0}^m C_i \omega_1^{m-i} \omega_2^i = \sum_{i=0}^m C_i z^{m-i} \varphi(z)^m$$

for some constants C_i .

Let Ψ be a meromorphic function on \mathcal{H}^* corresponding to an element Ψ_X in $K(X)$ satisfying the *parabolic residue conditions* with respect to $S^m(\Lambda^2)$ in the sense of [7, Definition 3.20]. We shall denote the space of such meromorphic functions by $\mathcal{M}_P(S^m(\Lambda^2))$, and for $\Psi \in \mathcal{M}_P(S^m(\Lambda^2))$ denote by $(S^m(\Lambda^2), \Psi)$ the nonhomogeneous differential equation

$$S^m(\Lambda^2) f = \Psi.$$

Let $\mathcal{D}(S^m(\Lambda^2))$ be the space of all such nonhomogeneous equations. If f_Ψ is a solution of the nonhomogeneous equation $(S^m(\Lambda^2), \Psi) \in \mathcal{D}(S^m(\Lambda^2))$, then it is known that the function

$$\xi_\Psi(z) = \frac{d^{m+1}}{dz^{m+1}} \left(\frac{f_\Psi(z)}{\varphi(z)^m} \right) \quad (5)$$

is an automorphic form of weight $m+2$ for Γ (cf. [7,8]).

DEFINITION 3.1. Let $\alpha \in \tilde{\Gamma}$, and let $T(\alpha)$ be the Hecke operator on the space $\mathcal{A}_{m+2}(\Gamma)$ of automorphic forms of weight $m+2$ for Γ described in (3).

(i) The Hecke operator $T_{\mathcal{M}}(\alpha)$ on $\mathcal{M}_P(S^m(\Lambda^2))$ is the map

$$T_{\mathcal{M}}(\alpha) : \mathcal{M}_P(S^m(\Lambda^2)) \rightarrow \mathcal{M}_P(S^m(\Lambda^2))$$

such that for each $\Psi \in \mathcal{M}_P(S^m(\Lambda^2))$, the automorphic form $\xi_{T_{\mathcal{M}}(\alpha)\Psi}$ corresponding to $T_{\mathcal{M}}(\alpha)\Psi$ is equal to the image $T(\alpha)\xi_\Psi$ of the automorphic form corresponding to Ψ under $T(\alpha)$.

(ii) The Hecke operator $T_{\mathcal{D}}(\alpha)$ on $\mathcal{D}(S^m(\Lambda^2))$ is the map $T_{\mathcal{D}}(\alpha) : \mathcal{D}(S^m(\Lambda^2)) \rightarrow \mathcal{D}(S^m(\Lambda^2))$ given by

$$T_{\mathcal{D}}(\alpha)(S^m(\Lambda^2), \Psi) = (S^m(\Lambda^2), T_{\mathcal{M}}(\alpha)\Psi).$$

THEOREM 3.2. Let $S^m(\Lambda^2)$ be as above, and let $T_{\mathcal{D}}(\alpha)$ and $T_{\mathcal{M}}(\alpha)$ be the Hecke operators associated to $\alpha \in \tilde{\Gamma}$ described in Definition 3.1. If $\Gamma\alpha\Gamma = \coprod_{\nu=1}^d \Gamma\alpha_\nu$ is the decomposition of the double coset $\Gamma\alpha\Gamma$ as in (2), then we have

$$T_{\mathcal{D}}(\alpha)(S^m(\Lambda^2), \Psi) = (S^m(\Lambda^2), T_{\mathcal{M}}(\alpha)\Psi)$$

for all $(S^m(\Lambda^2), \Psi) \in \mathcal{D}(S^m(\Lambda^2))$, where

$$(T_{\mathcal{M}}(\alpha)\Psi)(z) = \det(\alpha)^{m/2} \sum_{\nu=1}^d \det(\alpha_\nu)^{m/2+1} \left(\frac{\varphi(\alpha_\nu z)}{\varphi(z)} \right)^{m+2} \left(\frac{W_{\Lambda^2}(z)}{W_{\Lambda^2}(\alpha_\nu z)} \right)^{m+1} \frac{\Psi(\alpha_\nu z)}{j(\alpha_\nu, z)^{m+2}}$$

for all $z \in \mathcal{H}$. ■

PROOF. By [7, Theorem 3 bis. 5] the automorphic form ξ_Ψ of weight $m+2$ for Γ in (5) associated to $(S^m(\Lambda^2), \Psi)$ can be written as

$$\xi_\Psi(z) = (-1)^{m+1} \frac{\varphi(z)^{m+2} \Psi(z)}{W_{\Lambda^2}(z)^{m+1}}, \quad (6)$$

where $W_{\Lambda^2}(z)$ is the Wronskian for the differential operator Λ^2 . However, since ξ_Ψ is an automorphic form of weight $m+2$ for Γ , by (1), (3), and (5), we have

$$\begin{aligned} (T(\alpha)\xi_\Psi)(z) &= \det(\alpha)^{m/2} \sum_{\nu=1}^d (\xi_\Psi|_{m+2}\alpha_\nu) \\ &= \det(\alpha)^{m/2} \sum_{\nu=1}^d \det(\alpha_\nu)^{m/2+1} j(\alpha_\nu, z)^{-m-2} \xi_\Psi(\alpha_\nu z) \\ &= (-1)^{m+1} \det(\alpha)^{m/2} \sum_{\nu=1}^d \det(\alpha_\nu)^{m/2+1} j(\alpha_\nu, z)^{-m-2} \frac{\varphi(\alpha_\nu z)^{m+2} \Psi(\alpha_\nu z)}{W_{\Lambda^2}(\alpha_\nu z)^{m+1}}. \end{aligned} \quad (7)$$

On the other hand, by Definition 3.1(i) we have

$$(T(\alpha)\xi_\Psi)(z) = (-1)^{m+1} \frac{\varphi(z)^{m+2}(T_{\mathcal{M}}(\alpha)\Psi)(z)}{W_{\Lambda^2}(z)^{m+1}}. \quad (8)$$

Thus from (7) and (8) we obtain

$$\begin{aligned} (T_{\mathcal{M}}(\alpha)\Psi)(z) &= (-1)^{m+1} \frac{W_{\Lambda^2}(z)^{m+1}}{\varphi(z)^{m+2}} \\ &\quad \times (-1)^{m+1} \det(\alpha)^{m/2} \sum_{\nu=1}^d \det(\alpha_\nu)^{m/2+1} j(\alpha_\nu, z)^{-m-2} \frac{\varphi(\alpha_\nu z)^{m+2} \Psi(\alpha_\nu z)}{W_{\Lambda^2}(\alpha_\nu z)^{m+1}}. \end{aligned}$$

Hence the theorem follows from this and Definition 3.1(ii). ■

4. PARABOLIC COHOMOLOGY

Let $\Gamma \subset SL(2, \mathbb{R})$ be as in Section 3, and let $S^m(\mathbb{C}^2)$ be the m^{th} symmetric power of \mathbb{C}^2 . Then the natural action of $SL(2, \mathbb{R})$ on \mathbb{C}^2 induces a representation of Γ on $S^m(\mathbb{C}^2)$. Let $H_P^1(\Gamma, S^m(\mathbb{C}^2))$ be the parabolic cohomology of Γ with coefficients in $S^m(\mathbb{C}^2)$ (see, e.g., [6,7]). Then for each $\alpha \in \tilde{\Gamma}$, the Hecke operator $T_H(\alpha)$ on $H_P^1(\Gamma, S^m(\mathbb{C}^2))$ can be defined as follows (see [6,9,10] for details). Let $\Gamma\alpha\Gamma = \coprod_{\nu=1}^d \Gamma\alpha_\nu$ be the decomposition of the double coset $\Gamma\alpha\Gamma$. Then for each $\gamma \in \Gamma$, there is a permutation of $\{1, \dots, d\}$ sending ν to $\nu \cdot \gamma$ such that $\Gamma\alpha_\nu\gamma = \Gamma\alpha_{\nu \cdot \gamma}$ for $1 \leq \gamma \leq d$. Thus we have d maps $\rho_\nu : \Gamma \rightarrow \Gamma$ such that $\alpha_\nu\gamma = \rho_\nu(\gamma)\alpha_{\nu \cdot \gamma}$. If $[v] \in H_P^1(\Gamma, S^m(\mathbb{C}^2))$ is represented by a 1-cocycle $v : \Gamma \rightarrow S^m(\mathbb{C}^2)$, then $T_H(\alpha)[v]$ is an element of $H_P^1(\Gamma, S^m(\mathbb{C}^2))$ represented by the 1-cocycle $v' : \Gamma \rightarrow S^m(\mathbb{C}^2)$ given by

$$v'(\gamma) = \sum_{\nu=1}^d \det(\alpha_\nu) \cdot \alpha_\nu^{-1} \cdot v(\rho_\nu(\gamma))$$

for all $\gamma \in \Gamma$.

THEOREM 4.1. *Let $\mathcal{M}(\Gamma)$ be the space of meromorphic functions on \mathcal{H}^* that are liftings of elements of $K(X)$. Then there is a canonical isomorphism*

$$\frac{\mathcal{M}_P(S^m(\Lambda^2))}{S^m(\Lambda^2)(\mathcal{M}(\Gamma))} \cong H_P^1(\Gamma, S^m(\mathbb{C}^2)), \quad (9)$$

where $S^m(\Lambda^2)(\mathcal{M}(\Gamma))$ is the space of functions of the form $S^m(\Lambda^2)(h)$ with $h \in \mathcal{M}(\Gamma)$.

PROOF. This follows from [7, Theorem 5.7]. ■

THEOREM 4.2. *Given $\alpha \in \tilde{\Gamma}$ the Hecke operators $T_{\mathcal{M}}(\alpha)$ and $T_H(\alpha)$ are compatible with respect to the canonical isomorphism (9).*

PROOF. Let ξ_Ψ be the automorphic form of weight $m+2$ for Γ determined by $\Psi \in \mathcal{M}_P(S^m(\Lambda^2))$ as in (6). Then by [7, Theorem 5.7] the space $\mathcal{M}_P(S^m(\mathbb{C}^2))/S^m(\Lambda^2)(\mathcal{M}(\Gamma))$ is isomorphic to the space of automorphic forms of the form ξ_Ψ modulo the automorphic forms associates to the elements of $S^m(\Lambda^2)(\mathcal{M}(\Gamma))$. From the condition (i) in Definition 3.1 it follows that under this isomorphism the Hecke operators $T_{\mathcal{M}}(\alpha)$ and $T(\alpha)$ compatible. However the compatibility of $T(\alpha)$ and $T_H(\alpha)$ is also known by [6, Section 6.3]. Hence $T_{\mathcal{M}}(\alpha)$ and $T_H(\alpha)$ are compatible with respect to the isomorphism (9). ■

5. CONCLUDING REMARKS

As is mentioned in Section 1, the linear differential equations considered in Section 3 are closely related to automorphic forms, so it is natural to believe that there should be an analogue of the Hecke theory for such differential equations. This paper is an initial step for establishing such a theory.

REFERENCES

1. A. Poincaré, *Papers on Fuchsian Functions*, Springer-Verlag, New York, (1985).
2. A. Venkov, *Spectral Theory of Automorphic Functions and Its Applications*, Kluwer, Dodrecht, (1990).
3. T. Miyake, *Modular Forms*, Springer-Verlag, New York, (1989).
4. A. Andrianov, *Quadratic Forms and Hecke Operators*, Springer-Verlag, New York, (1987).
5. A. Krieg, Hecke algebras, In *Mem. Amer. Math. Soc.*, Volume 435, Amer. Math. Soc., Providence, RI, (1990).
6. H. Hida, *Elementary Theory of L-Functions and Eisenstein Series*, Cambridge University Press, (1993).
7. P. Stiller, Special values of Dirichlet series, monodromy, and the periods of automorphic forms, In *Mem. Amer. Math. Soc.*, Volume 299, Amer. Math. Soc., Providence, RI, (1984).
8. M.H. Lee, Mixed automorphic forms and differential equations, *Internat. J. Math. and Math. Sci.* **13**, 661–668 (1990).
9. M. Kuga, W. Parry and C. Sah, Group cohomology and Hecke operators, In *Manifolds and Lie Groups*, (Edited by J. Hano, A. Morimoto, S. Murakami, K. Okamoto and H. Ozeki), pp. 223–266, Birkhäuser, Boston.
10. Y. Rhie and G. Whaples, Hecke operators in cohomology of groups, *J. Math. Soc. Japan* **22**, 431–442 (1970).